A Trig Identities

A.1. All of the trigonometric functions of an angle θ can be constructed geometrically in terms of a unit circle centered at origin as shown in Figure 85.

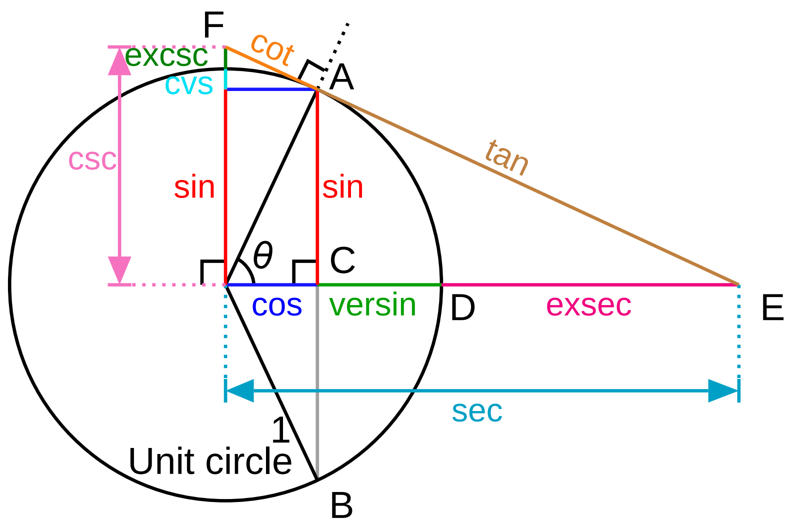


Figure 85: Trigonometric functions on a unit circle.

A.2. Cosine function

(a) Is an even function: $\cos(-x) = \cos(x)$.

(b) $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$.

(c) Sum formula:

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y. \tag{91}$$

(d) Product-to-Sum Formula:

$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right).$$

(e) $\cos^n x = \begin{cases} \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \cos\left((n-2k)\,x\right), & \text{odd } n \geq 1 \\ \frac{1}{2^n} \left(\sum_{k=0}^{\frac{n}{2}-1} 2\binom{n}{k} \cos\left((n-2k)\,x\right) + \binom{n}{\frac{n}{2}} \right), & \text{even } n \geq 2 \end{cases}$

(f) Any two real numbers a, b can be expressed in terms of cosine and sine with the same amplitude and phase:

$$(a, b) = (A\cos(\phi), A\sin(\phi)), \tag{92}$$

where $A = \sqrt{a^2 + b^2}$ and $\phi = \text{atan2}(b, a)$. This is simply the polar-coordinates from of the point (a, b) on Cartesian coordinates. Note that atan2 is the four quadrant inverse tangent which takes the signs of a and b into account to determine in which quadrant ϕ lies.

A.3. The complex exponential function $e^{j\theta}$

(a) As a function of θ, $e^{j\theta}$ is periodic with period 2π.

(b) ***Euler's formula***: $\boxed{e^{j\theta} = \cos\theta + j\sin\theta.}$

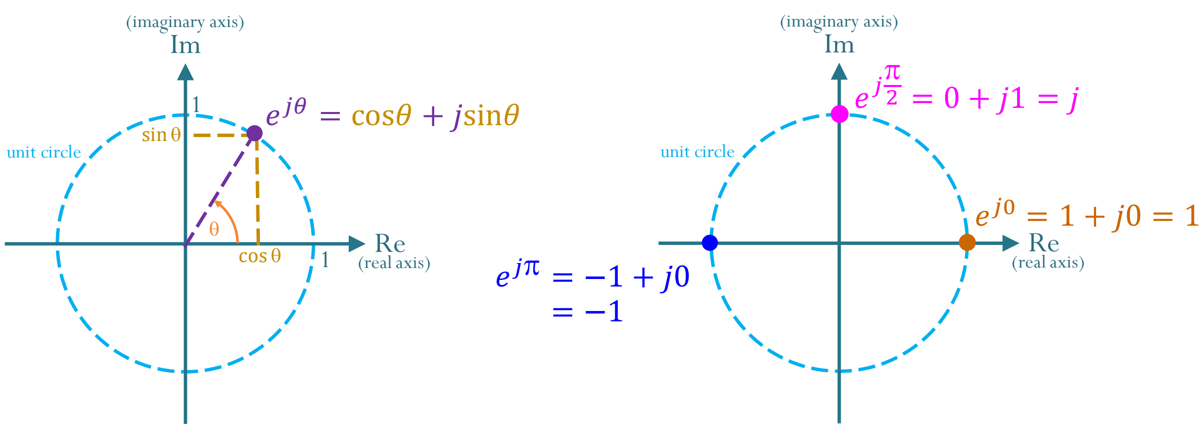


Figure 86: Euler's Formula on the Complex Plane

$$\cos(\theta) = \text{Re}\left\{e^{j\theta}\right\} = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$\sin(\theta) = \text{Im}\left\{e^{j\theta}\right\} = \text{Re}\left\{-je^{j\theta}\right\} = \text{Re}\left\{+\frac{1}{j}e^{j\theta}\right\} = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right).$$

(c) Any complex number $z = x + jy$ can be expressed as

$$z = \sqrt{x^2 + y^2}e^{j\phi} = |z|e^{j\phi},$$

where $\phi = \text{atan2}(b, a)$.

- $z^t = |z|^t e^{j\phi t}$.

(d) Using the Euler's formula, we can rewrite a linear combination of two cosines as a single cosine:

$$\begin{aligned} a_1 \cos(\theta_1) + a_2 \cos(\theta_2) &= a_1 \operatorname{Re}\left\{e^{j\theta_1}\right\} + a_2 \operatorname{Re}\left\{e^{j\theta_2}\right\} \\ &= \operatorname{Re}\left\{a_1 e^{j\theta_1} + a_2 e^{j\theta_2}\right\}. \end{aligned}$$

The sum $a_1 e^{j\theta_1} + a_2 e^{j\theta_2}$ is a sum of two complex numbers and hence is another complex number. Suppose the polar form of this number is $ae^{j\theta}$. Then,

$$a_1 \cos(\theta_1) + a_2 \cos(\theta_2) = \operatorname{Re}\left\{ae^{j\theta}\right\} = a\cos\theta.$$

(e) Using the Euler's formula, we can rewrite linear combination of cosine and sine of the same argument as a single cosine by

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta - \phi),$$

where $\phi = \text{atan2}(b, a)$. To see this, note that

$$a\cos\theta + b\sin\theta = \operatorname{Re}\left\{ae^{j\theta}\right\} + \operatorname{Re}\left\{-jbe^{j\theta}\right\} = \operatorname{Re}\left\{(a - jb)\, e^{j\theta}\right\}.$$

We know that $a + jb = \sqrt{a^2+b^2}e^{j\phi}$ where $\phi = \text{atan2}(b, a)$. So, $a - jb = (a + jb)^* = \sqrt{a^2+b^2}e^{-j\phi}$. Hence,

$$a\cos\theta + b\sin\theta = \operatorname{Re}\left\{\sqrt{a^2+b^2}e^{-j\phi}e^{j\theta}\right\}$$

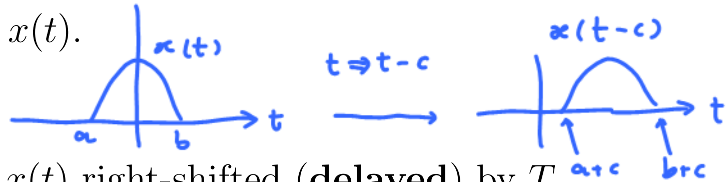
Another way to see this is to re-express the two real numbers a, b using (92) and then use (91).

(f) More relations involving sin and cos.

- $e^{jAt} + e^{jBt} = 2e^{j\frac{A+B}{2}t}\cos\left(\frac{A-B}{2}\right)$.
- $e^{jAt} - e^{jBt} = 2je^{j\frac{A+B}{2}t}\sin\left(\frac{A-B}{2}\right)$
- $\frac{e^{jAt}-e^{jBt}}{e^{jCt}-e^{jDt}} = e^{j\frac{(A+B)-(C+D)}{2}t}\frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C-D}{2}\right)}$.

B Time Manipulation

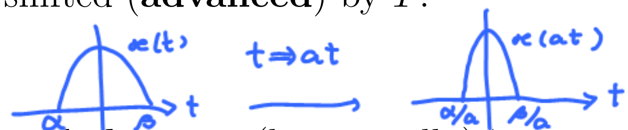
B.1. Consider a function of time $x(t)$.



(a) **Time shifting**:

(i) When $T > 0$, $x(t - T)$ is $x(t)$ right-shifted (**delayed**) by T.

(ii) When $T < 0$, $x(t - T)$ is $x(t)$ left-shifted (**advanced**) by T.



(b) **Time scaling** (horizontal scaling):

(i) When $0 < a < 1$, $x(at)$ is $x(t)$ **expanded** in time (horizontally) by a factor of $\frac{1}{a}$.

(ii) When $a > 1$, $x(at)$ is $x(t)$ **compressed** in time (horizontally) by a factor of a.

- Note that the signal remains anchors at $t = 0$. In other words, the signal at $t = 0$ remains unchanged.

(c) **Time inversion** (or folding):

- $x(t)$ is the mirror image of $x(t)$ about the vertical axis.

Example B.2. A function of the form $x(mt + c)$ can be viewed as

(a) $x((mt) - (-c))$: First right-shift $x(t)$ by $-c$. (Equivalently, left-shift $x(t)$ by c.) Then scale horizontally by a factor of $\frac{1}{m}$.

(b) $x\left(m\left(t - \left(-\frac{c}{m}\right)\right)\right)$: First scale $x(t)$ horizontally by a factor of $\frac{1}{m}$. Then, right-shift by $-\frac{c}{m}$.

These two approaches are illustrated in Figure 87.

Alternatively, it may be easier to look at where the key points in the plot will show up in the new plot. For example, let's consider the leftmost point in the original plot of $x(t)$. Note that it occurs when the argument of $x(t)$ is a. So, in the plot of $x(mt + c)$, it will occur at t such that $mt + c = a$. Therefore, it will be at $t = \frac{a-c}{m}$.

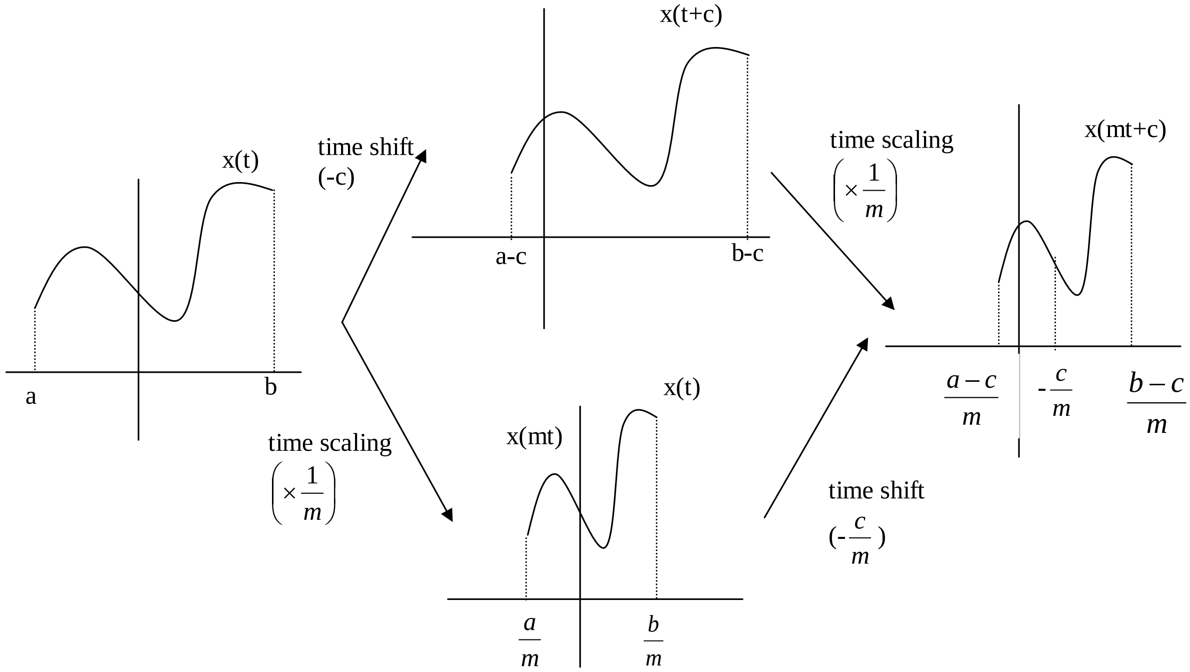


Figure 87: Two approaches for drawing $x\,(mt + c)$.

Example B.3. Consider the function $x(t)$ given in Figure 88.

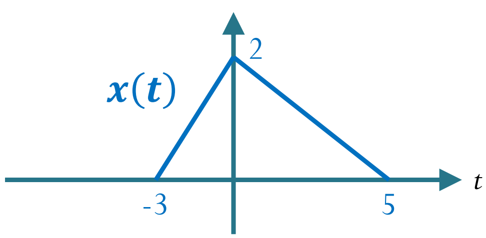


Figure 88: The function $x(t)$ used in Example B.3.

(a) Find the area under $x(t)$.
Solution: See the top part of Figure 89.

(b) Plot and find the area under the new function $x(-t)$.
Solution: See Figure 89.

(c) Plot and find the area under $x(2t)$.
Solution: See Figure 90 for derivation of the plot then see Figure 91 for calculation of the area.

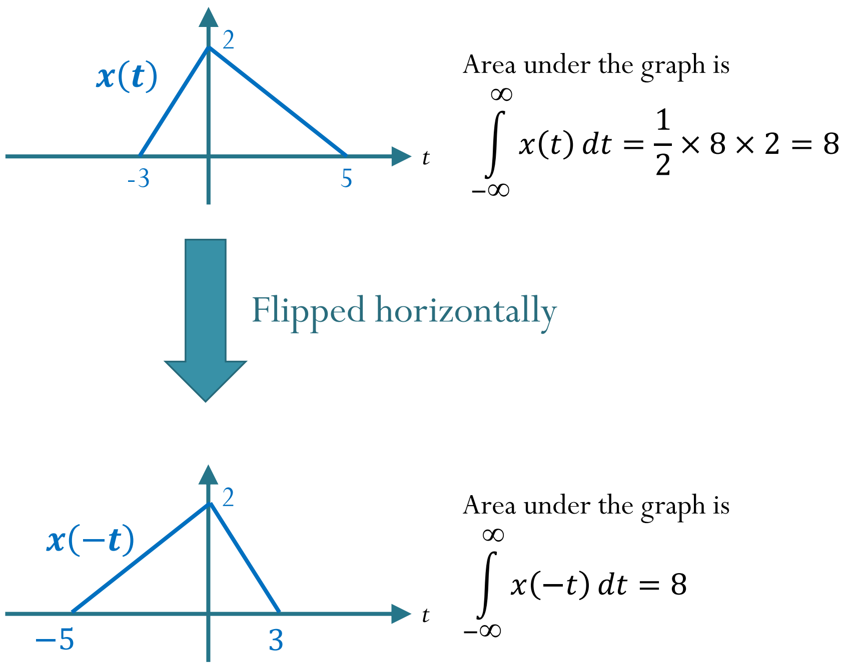


Figure 89: Example of time inversion.

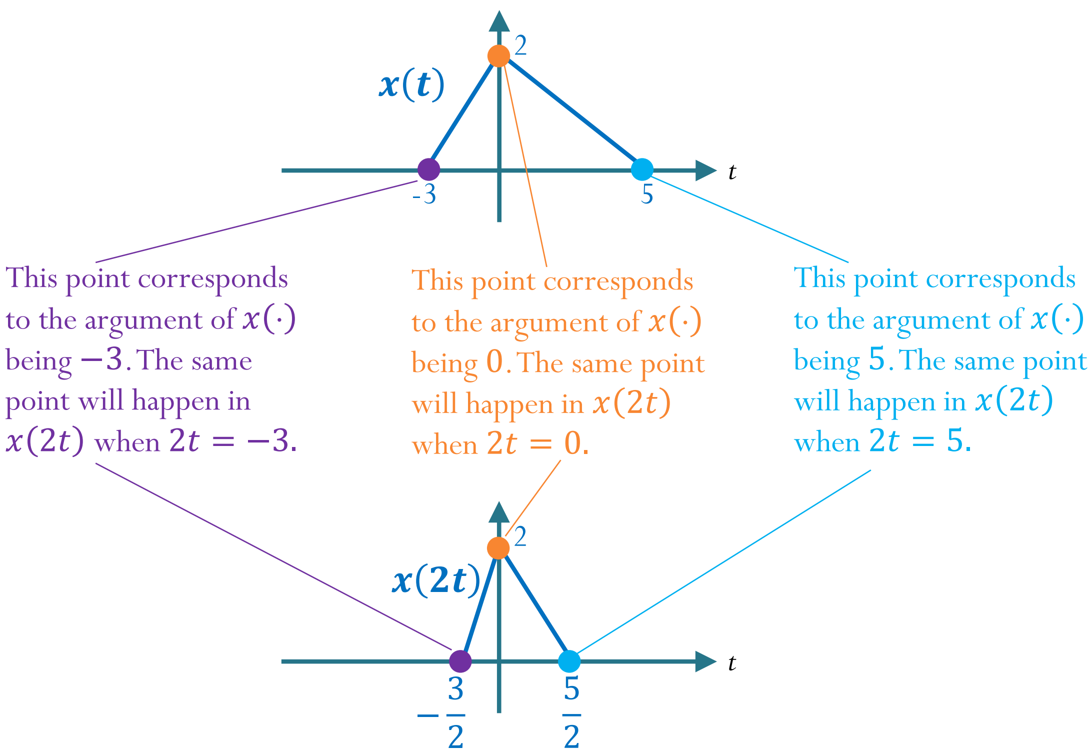


Figure 90: Example of time scaling.

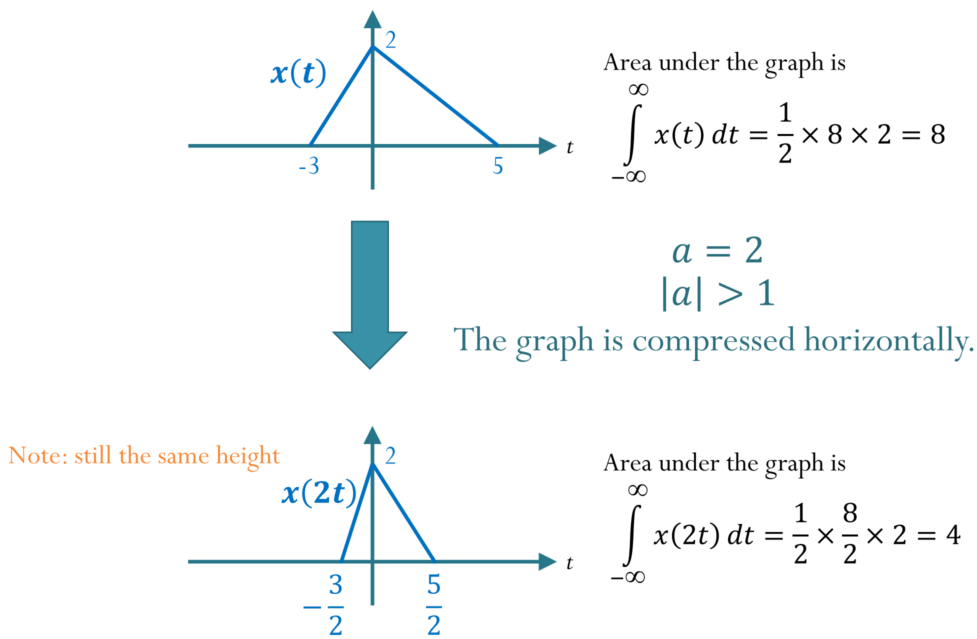


Figure 91: Calculating the area when the function is scaled horizontally.

B.4. Figure 92 shows the delta functions as limits of rectangular functions whose width are compressed to 0 while the areas are kept constant.

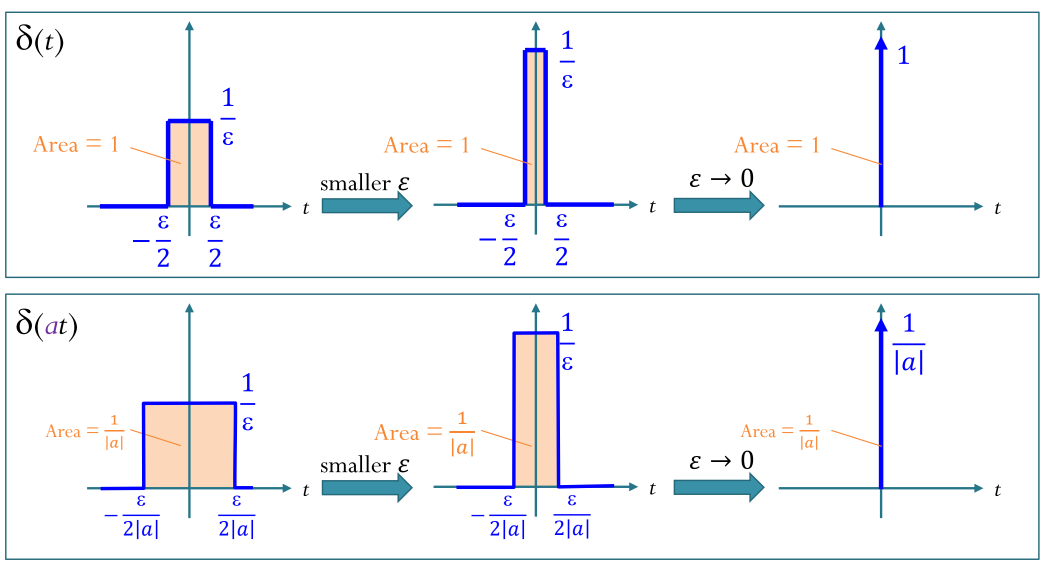


Figure 92: Delta functions as limits of rectangular functions.Here, $a = \frac{1}{2}$.

C Linear System

Definition C.1. System:

Definition C.2. A **linear system** is a system whose output is linearly related (or directly proportional) to its input[28]. In particular, when we says that the input and output are linearly related, we mean they need to satisfies two properties:

(a) Homogeneous (Scaling): If the input is multiplied by a constant k, then we should observed that the output is also multiplied by k.

(b) Additive: If the inputs are summed then the output are summed.

Example C.3. Is the function $f(x) = x^2 + 1$ linear?

Example C.4. Is the function $f(x) = 3x + 1$ linear?

C.5. Any **one-dimensional linear function** can be written in the form

$$y = ax$$

for some constant a.

[28]The input and output are sometimes referred to as cause and effect, respectively.

- For a system, we may call it a **single-input single-output (SISO)** system.
- In radio it is the use of only one antenna both in the transmitter and receiver.

C.6. Any **multi-dimensional linear function** can be written in the form

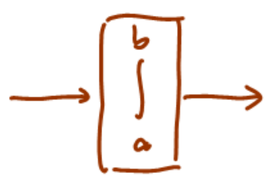
$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

for some matrix $\mathbf{A}$.

- For a system, when both m and n are greater than one, we may call it a **multiple-input multiple-output system (MIMO)** system.

- When $m = n = 1$, we are back to the one-dimensional case in C.5.

More examples for linearity

① 

$$\int_a^b \left(c_1 f_1(x) + c_2 f_2(x)\right) dx = c_1 \int_a^b f_1(x)\, dx + c_2 \int_a^b f_2(x)\, dx$$

② 

$$\mathcal{F}\{c_1 g_1 + c_2 g_2\} = c_1 \mathcal{F}\{g_1\} + c_2 \mathcal{F}\{g_2\}$$